O and N follows independently also from the fact that O and N form a *Steinerian couple*[1] on the cubic, *i.e.*, the points of tangency of two tangents from a point on the cubic (in this case the infinite point of C_3) to the same cubic. Like the cyclide itself, the cubic C_3 is anallagmatic with respect to O and N. In fact

$$OT \cdot OU = OW \cdot OV = 1; \qquad NV \cdot NU = NW \cdot NT = \frac{1 - 4c^2}{4c^2}.$$

In case of Ω_1 and Ω_4 the anallagmatic property reduces to symmetry with respect to these planes. In case of Ω_5 the anallagmatic constant (radius square of Ω_5) is -1.

A CURVE OF PURSUIT.

By F. V. MORLEY, New College, Oxford University.

(Read before the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America, May 15, 1920.)

The curve of pursuit is one of that class of problems so entertainingly described by Professor David Eugene Smith, which in their travel through the centuries have preserved traces of the times of their proposers. The problem in one dimension, of the pursuer following the pursued in line, is common since the time of Zeno's paradox[2]; but the curve of pursuit does not seem to have been studied till the 18th century. An attempt has been made to make Leonardo da Vinci responsible, among his other wealth of contributions, for the statement of the problem.[3] But although it is quite possible to read into Leonardo's passage the essence of the question, it is perhaps doubtful that he ever had a conscious formulation. And of necessity, careful consideration of the problem had to wait until the methods of the calculus were known.

At any rate, the problem of the curve of pursuit was stated by Bouguer in 1732.[4] Although the days of the buccaneers were numbered, it is characteristic of the times that he chose for his example a privateer and a merchant vessel. Bouguer considered only the simplest case, where the pursued point moves along a line, but in the same volume of the *Mémoires de l'Académie Royale des Sciences* is a generalization of the problem by the remarkable de Maupertuis. Since then the problem in various guises has appeared in texts and periodicals.[5] One simple variant, in which the pursued point moves along a circle and the pursuer starts from the center, was re-proposed by Professor A. S. Hathaway in this MONTHLY, *1920*, 31. It is this case which is considered in this paper.

[1] For the theory of Steinerian couples and quadruples on plane cubics see the author's *Introduction to Projective Geometry*, New York, 1905, pp. 197–204.

[2] See D. E. Smith, AMER. MATH. MONTHLY, Vol. 24, 1917, p. 64.

[3] Brocard, *Nouv. Corr. Math.*, Vol. 6, 1880, p. 211; cf. Loria, *Ebene Kurven*, 1902, p. 608.

[4] *Mémoires de l'Académie Royale des Sciences*, 1732.

[5] *E.g.*, *Math. Monthly* (ed. J. D. Runkle), 1, 1859, p. 249. There are also more elaborate papers such as "Sur les courbes de poursuite d'un cercle," by M. L. Dunoyer, *Nouv. Annales de Math.*, 4th series, Vol. 6, 1906, p. 193. [Compare page 91 of this issue.—EDITOR].

The problem suggested by Dr. Hathaway is illustrated in Fig. 1. A duck is swimming with constant speed around the edge of a circular pond; a dog starts from the center and swims always directly towards the duck, c (a constant) times as fast. Suppose that the radius of the pond is unity, and that at any instant the duck has traveled over an arc θ from its starting point; then the dog at the same instant will have traveled a distance s along its curved path, where

$$s = c\theta.$$

If as shown in the figure we use p and ω as normal coördinates of the line joining dog and duck (which is always tangent to the dog's path), the radius of curvature of the dog's path will be

$$p + \frac{d^2p}{d\omega^2} = \frac{ds}{d\omega} = c\frac{d\theta}{d\omega}.$$

Now from the figure

$$p = \sin(\omega - \theta)$$

or

$$\theta - \omega = -\sin^{-1} p,$$

so that

$$\frac{d\theta}{d\omega} = 1 - \frac{\dfrac{dp}{d\omega}}{\sqrt{1 - p^2}}.$$

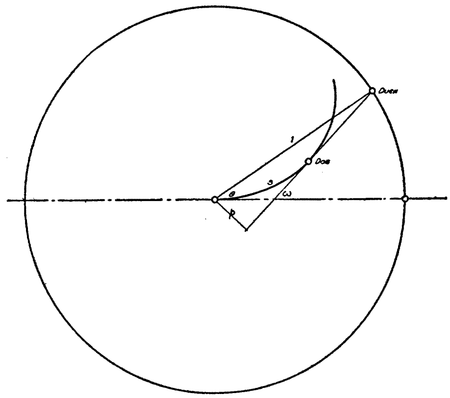

FIG. 4.

The differential equation then becomes

$$p + \frac{d^2p}{d\omega^2} = c\left[1 - \frac{\dfrac{dp}{d\omega}}{\sqrt{1 - p^2}}\right].$$

In this we make the substitution $dp/d\omega = u$. Then

$$\frac{d^2p}{d\omega^2} = u\frac{du}{dp}$$

and

$$p + u\frac{du}{dp} = c\left[1 - \frac{u}{\sqrt{1 - p^2}}\right].$$

Now let $1 - u^2 - p^2 = 1 - r^2 = v$, where $-v$ is the *power* of the point with respect to the circle; then

$$\frac{dv}{dp} = -2\left[u\frac{du}{dp} + p\right]$$

and finally[1]

[1] The equations of the normal and tangent to the dog's path may be written

$$x \cos\omega + y \sin\omega = \cos(\omega - \theta) - p$$

and

$$x \sin\omega - y \cos\omega = \sin(\omega - \theta),$$

$$\left(\frac{dv}{dp}\right)^2 + 4c\frac{dv}{dp} + \frac{4c^2v}{1-p^2} = 0.$$

This is the differential equation giving the dog's path, innocent enough in looks, but apparently not integrable by the ordinary methods. An approximate solution may then be sought by integration in series, or by graphical methods. Let us consider the latter.

Writing the differential equation in the more familiar variables x and y and using these as rectangular coördinates, we have to discuss the following:

$$\left(\frac{dy}{dx}\right)^2 + 4c\frac{dy}{dx} + 4c^2\frac{y}{1-x^2} = 0.$$

In the first place, at every point of the plane two directions are determined by this equation; namely, the roots of the quadratic in dy/dx. The locus of all points for which either of these directions is constant and equal to m, is given by

$$m^2 + 4cm + 4c^2\frac{y}{1-x^2} = 0.$$

This is a family of parabolas determined by the parameter m, and all passing through the points $(\pm 1, 0)$.

The equation thus determines two families of integral curves, such that each member of each family cuts every parabola in a definite and determinable direction. In order to pick out the particular integral curves in which we are interested, let us notice the initial conditions of the problem. When the dog is at the center of the circle, $p = 0$ and $v = 1$, so that in the new notation we want the particular integral curves which pass through the point $x = 0$, $y = 1$. And what we want to find is the value of p (or x) when v (or y) is 0. We therefore wish to trace the integral curves from the point $(0, 1)$ as far as the x-axis, and to determine their intercepts.

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where ρ denotes the distance between the dog and duck. These equations are satisfied by the coordinates of the dog's position and may be regarded as expressing these coordinates in terms of ω, θ and ρ.

If we differentiate and divide by $d\theta$, remembering that $dy = dx \tan \omega$, and $dx \sec \omega = ds = cd\theta$, we shall get equations which reduce to

$$\frac{d\rho}{d\theta} = \sin(\omega - \theta) - c \tag{1}$$

and

$$\rho\frac{d\omega}{d\theta} = \cos(\omega - \theta). \tag{2}$$

These are the equations derived by Professor Hathaway in his solution of Problem 2801 given on pages 93–97, θ, $\omega - \theta$, ρ and c being the same as his s, θ, r and k.

Now $v = \rho[2\cos(\omega - \theta) - \rho]$, and $p = \sin(\omega - \theta)$, and these equations by aid of (1) and (2) lead directly to Mr. Morley's differential equation in p and v.

Dunoyer (l. c.) also derives equations (1) and (2) as expressing the components of the dog's velocity in the direction of the duck and at right angles to this direction.—Editor.

Let us then draw a few of the family of parabolas, lying between the x-axis and the parabola through (0, 1). The parabola through (0, 1) will be

$$y = 1 - x^2$$

and the directions at every point of this parabola will be the roots of

$$m^2 + 4cm + 4c^2 = 0$$

namely,

$$m_1 = -2c, \qquad \text{and} \qquad m_2 = -2c.$$

For this parabola the directions coincide.

Another parabola would be

$$y = \tfrac{3}{4}(1 - x^2),$$

cutting the y-axis at (0, $\frac{3}{4}$). Here the directions are given by

$$m^2 + 4cm + 3c^2 = 0,$$

and are

$$m_1 = -3c, \qquad \text{and} \qquad m_2 = -c;$$

and so for as many parabolas as we care to draw. Finally, on the limiting parabola $y = 0$ the directions are given by

$$m^2 + 4cm = 0,$$

and are

$$m_1 = -4c, \qquad \text{and} \qquad m_2 = 0.$$

Now to solve any particular case we have to attach to each parabola directions according to the value of c, and then the two particular integral curves in which we are interested (one will go with m_1 and the other with m_2) may be plotted without difficulty and to a considerable degree of accuracy, using the methods given by Runge.[1]

Let us for instance in Fig. 2 construct the integral curves for $c = 3$. For the curve defined by m_1 we start by drawing a line from (0, 1) with a slope of -6. Mark a point on this line about half-way between the two outer parabolas, and through that point draw a line with a slope of -8, which is the direction appropriate to the second parabola. Repeat the process, using a point on this line half-way between the second and third parabolas, etc. These construction lines are not shown in Fig. 2, but by continuing the process an approximation to the integral curve for m_1 is obtained. This is dotted in the figure, and in this case cuts the x-axis at $x = 0.10$. The curve for m_2 is drawn by the same process, though here the first approximation may not be sufficiently accurate. A second approximation may be obtained by successive differentiation and integration by graphical methods, as in Runge. When drawn, as in the full line of Fig. 2, the curve for m_2 cuts the x-axis at $x = 1$.

[1] C. Runge, *Graphical Methods*, Columbia University Press, 1912, p. 120.

In order to see the meaning of these solutions, let us draw in Fig. 3 the actual curve of pursuit. This may be very accurately drawn by a bracketing method. That is, by marking a series of small equal steps for the duck around the circle; one curve may be drawn where the dog is successively directed towards the

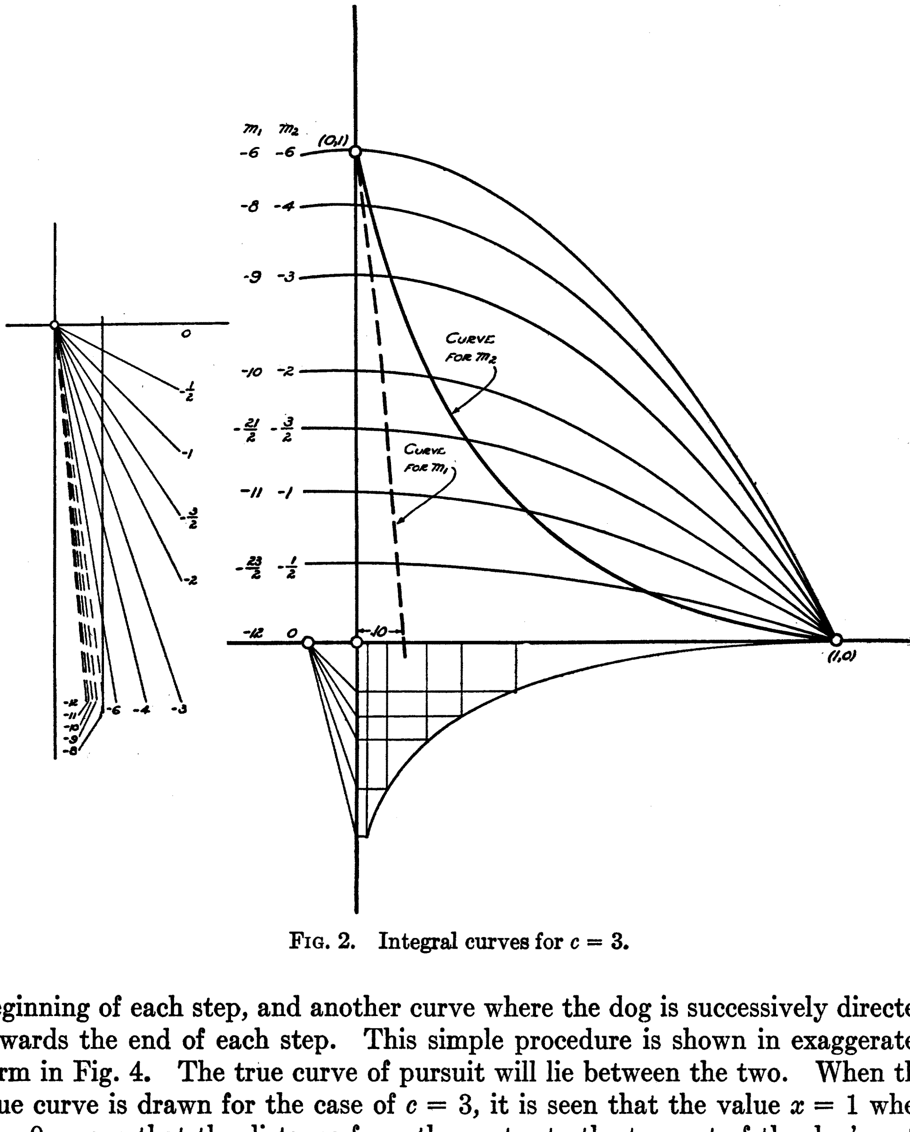

FIG. 2. Integral curves for $c = 3$.

beginning of each step, and another curve where the dog is successively directed towards the end of each step. This simple procedure is shown in exaggerated form in Fig. 4. The true curve of pursuit will lie between the two. When the true curve is drawn for the case of $c = 3$, it is seen that the value $x = 1$ when $y = 0$ means that the distance from the center to the tangent of the dog's path is 1 wherever the path cuts the circle. In other words, the dog comes up to the duck tangent to the circle. But if we had been tracing the dog's curve backwards from the center, as if he had been in flight instead of in pursuit, he would have reached the circle at a point where the tangent to his path is distant (by

measurement of Fig. 3) 0.093 from the center. The first approximation of the curve for m_1 thus checks reasonably with the curve of flight as drawn.

But let us notice there may be cases of difficulty in drawing the integral curves. In any case where c is less than 2, for instance, Runge's method breaks down, if followed blindly. If we follow the curve for m_2 we come to a point on one of the parabolas where the slope of the parabola is itself equal to m_2. The integral curve cannot there simply cross the parabola, yet it cannot turn upward. The presumption is that there the curve has a flex, and this we shall have to test.

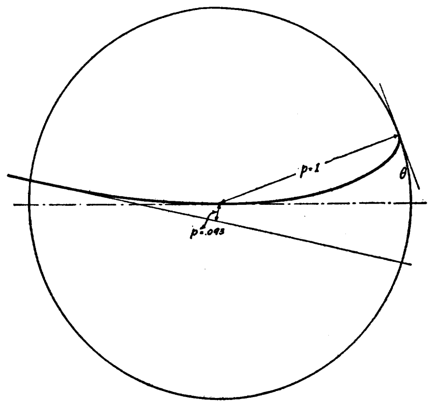

FIG. 3. Curve of pursuit, $c = 3$, $\theta = .058$.

First let us find the locus of those points where the direction assigned by the differential equation is the same as the slope of the parabola through that point. Call for brevity

$$\mu = \frac{y}{1 - x^2}.$$

The slope of the parabola is

$$\frac{dy}{dx} = -2\mu x = m.$$

Then

$$4\mu^2 x^2 - 8c\mu x + 4c^2\mu = 0.$$

If $\mu \neq 0$,

$$\mu x^2 - 2cx + c^2 = 0,$$

$$x^2 y - c(2x - c)(1 - x^2) = 0,$$

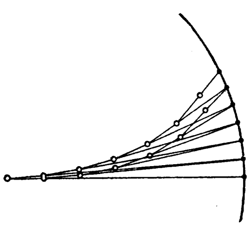

FIG. 4. Details of bracketing method for drawing the curve of pursuit.

so that the cubic

$$y = \frac{c}{x^2}(2x - c)(1 - x^2)$$

is the locus of points for which the direction assigned is equal to the slope. This will cut the axis at $c/2$ and ± 1, and may be easily drawn.

Now the integral curve will have flexes where

$$\frac{d^2y}{dx^2} = 0, \qquad \text{or} \qquad \frac{dm}{dx} = 0.$$

But

$$2m\frac{dm}{dx} + 4c\frac{dm}{dx} + 4c^2\left[\frac{(1 - x^2)m + 2xy}{(1 - x^2)^2}\right] = 0.$$

Hence, there are flexes, when

$$m = -\frac{2xy}{1 - x^2} = -2\mu x,$$

and this leads to the same cubic given in the preceding paragraph. Thus a flex does occur whenever the integral curve cuts the cubic.

Cusps will occur on the integral curve when

$$\frac{d^2y}{dx^2} = \frac{dm}{dx} = \infty$$

or when

$$m = -2c$$

and that is, on the parabola

$$y = 1 - x^2.$$

This additional information enables us to draw the integral curves with more ease and accuracy. There is very little difficulty about the integral curve for m_1, corresponding to the curve of flight, and we shall not consider this further. But the behavior of the curve for m_2 may be more complicated. This is the case in Fig. 5, drawn for $c = 1/2$. Fig. 6 shows the actual curve of pursuit drawn by the bracketing method. It is to be expected that the dog will pursue a path asymptotic to an inner concentric circle of radius $1/2$. But the integral curve of Fig. 5 shows very nicely just how the path approaches the asymptotic circle. Following down the curve from $(0, 1)$, there is first an intersection with the cubic, and a consequent flex, as shown. After this it cuts the cubic again, with another flex, and then meets the outside parabola at $x = 0.71$. Here it must have a cusp, which of necessity is of the rhamphoid type. Then it goes back, cutting the cubic twice more, and again meeting the outside parabola in a rhamphoid cusp at $x = 0.42$. The oscillations continue, growing smaller and smaller, and becoming asymptotic to the point where the cubic touches the outside parabola, namely $(1/2, 3/4)$. A glance at Fig. 6 shows that the cusps of Fig. 5 indicate the apses in the dog's path.

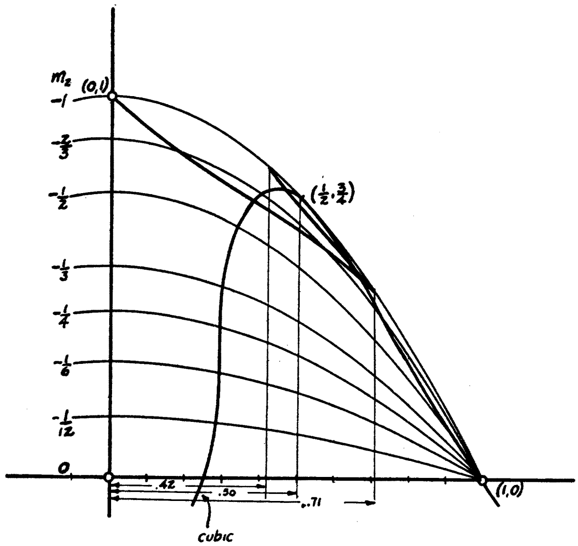

FIG. 5. Curve of pursuit, integral curves for $c = \frac{1}{2}$. Only curve for m_2 is drawn.

The results may then be summarized into three divisions, according to the value of c. When $c > 2$, the cubic does not interfere. When $1 < c < 2$, the cubic interferes, but does not affect the ultimate passage of the integral curve to $(1, 0)$. When $0 < c < 1$, the cubic touches the outside parabola at the point

$(c, 1 - c^2)$, and the integral curve becomes asymptotic to this point by a series of cusps. In the first two cases the dog overtakes the duck tangentially, though with complete indetermination of his future course; and in the last one fails.

But although the differential equation has been solved as accurately as desired, the answer to the problem, namely the distance traveled by dog or duck, has not yet been obtained. The equation tells how to draw the curve of pursuit, but does not tell its length. For this an approximate formula may be derived from experiment. By measurement of the figures we know the corresponding values:

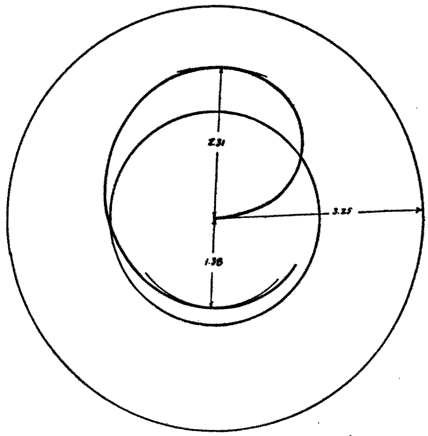

FIG. 6. Curve of pursuit, $c = \frac{1}{2}$; $p = (2.31/3.25) = 71$; $p = (1.38/3.25) = .42$.

c	θ	
less than 1	imaginary	
1	∞	
3/2	.136	(Figure not repro-
3	.058	duced.)
∞	0	

Suppose that at a hazard we set up the empirical formula

$$\theta = \frac{A}{\sqrt{c^2 - 1}} .$$

This is satisfied when $c = 1$, and when $c = \infty$, independently of A. This is imaginary like θ when $c < 1$. For $c = 3/2$, $A = .150$, and when $c = 3$, $A = .162$. As a first approximation we might then use the formula

$$\theta = \frac{0.156}{\sqrt{c^2 - 1}} .$$

The curve of pursuit forms a good problem to test graphical methods of solving a differential equation,[1] since here the actual curve can be drawn easily, and the accuracy of the graphical solutions tested. This comparison shows graphical methods to be very satisfactory when used with care on such an equation as the above, and would lend confidence to cases where the actual curve may not be so easy to draw.

[1] For an interesting and elementary account of the graphical treatment of differential equations, see H. Brodetsky, *Mathematical Gazette*, October, 1919, and January, March, and May, 1920.